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TWO VARIABLE HIGHER-ORDER DEGENERATE FUBINI POLYNOMIALS

DAE SAN KIM, GWAN-WOO JANG, HYUCK-IN KWON, AND TAEKYUN KIM

ABSTRACT. Fubini numbers (also called ordered Bell numbers) have been studied by several authors (see [2, 3, 4, 6, 8]). Recently, Kim-Kim studied the two variable Fubini polynomials and degenerate Fubini polynomials (see [6-8]). In this paper, we consider the higher-order two variable degenerate Fubini polynomials by using umbral calculus. We present several explicit formulas and recurrence relations for these polynomials. In addition, we express the higher-order two variable degenerate Fubini polynomials in terms of some families of special polynomials and vice versa.

1. Introduction

The two variable degenerate Fubini polynomials $F_{n,\lambda}^{(r)}(x;y)$ of order r are defined by

$$\left(\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n0}^{\infty}F_{n,\lambda}^{(r)}(x;y)\frac{t^{n}}{n!}, \tag{1.1}$$

where r is a positive integer and $\lambda \in \mathbb{R}$. In this paper, y will be an arbitrary but fixed real number so that $F_{n,\lambda}^{(r)}(x;y)$ are polynomials in x for each fixed y.

When $r=1$, $F_{n,\lambda}(x;y)=F_{n,\lambda}^{(1)}(x;y)$ are called two variable degenerate Fubini polynomials and they are introduced in [6] as a degenerate version of two variable Fubini polynomials in [2, 7, 8].

If $x=0$, $F_{n,\lambda}^{(r)}(y)=F_{n,\lambda}^{(r)}(0;y)$ and $F_{n,\lambda}^{(r)}=F_{n,\lambda}^{(r)}(1)=F_{n,\lambda}^{(r)}(0;1)$ are called the degenerate Fubini polynomials of order r and the degenerate Fubini numbers of order r, respectively.

Further, in the special case of $y=1$, $F_{n,\lambda}^{(r)}(x;1)$ are denoted by $Ob_{n,\lambda}^{(r)}(x)$ and called the degenerate ordered Bell polynomials; $F_{n,\lambda}^{(r)}(1)=F_{n,\lambda}^{(r)}(0;1)$ are also

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denoted by $Ob_{n,\lambda}^{(r)}$ and also called the degenerate ordered Bell numbers. Thus $Ob_{n,\lambda}^{(r)}(x)$ and $Ob_{n,\lambda}^{(r)}$ are respectively given by the generating functions

$$\left(\frac{1}{2-(1+\lambda t)^{\frac{1}{\lambda}}}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ob_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!},\tag{1.2}$$

$$\left(\frac{1}{2-(1+\lambda t)^{\frac{1}{\lambda}}}\right)^{r}=\sum_{n=0}^{\infty}Ob_{n,\lambda}^{(r)}\frac{t^{n}}{n!}.\tag{1.3}$$

In this paper, by using umbral calculus we would like to investigate the two variable higher-order degenerate Fubini polynomials and derive their properties, recurrence relations and some identities. Especially, we will express some well-known families of special polynomials as linear combinations of the two variable higher-order degenerate Fubini polynomials and vice versa.

2. Review on umbral calculus

The purpose of this paper is to use umbral calculus in order to study the two variable higher-order degenerate Fubini polynomials. Here we will go over some of the basic facts about umbral calculus. One may refer to [10] for a complete treatment of modern umbral calculus which is now a rigorous and fascinating area of mathematics, thanks to the effort of Tian-Carlo Rota and others.

Let $\mathbb{C}$ be the field of complex numbers. By $\mathcal{F}$ we denote the algebra of all formal power series in the variable t with the coefficients in $\mathbb{C}$:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\Big|a_k\in\mathbb{C}\right\}.$$

Let $\mathbb{P}=\mathbb{C}[x]$ denote the ring of polynomials in x with the coefficients in $\mathbb{C}$. Then, by $\mathbb{P}^*$ we indicate the vector space of all linear functionals on $\mathbb{P}$. For each $L\in\mathbb{P}^*$, and each $p(x)\in\mathbb{P}$, $<L|p(x)>$ denotes the action of the linear functional L on $p(x)$.

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$, we let $<f(t)|\cdot>$ denote the linear functional on $\mathbb{P}$ given by

$$<f(t)|x^n>=a_n,\ \ (n\geq 0).$$

For $L\in\mathbb{P}^*$, let $f_L(t)=\sum_{k=0}^{\infty}<L|x^k>\frac{t^k}{k!}\in\mathcal{F}$. Then $<f_L(t)|x^n>=<L|x^n>$, for all $n\geq 0$, and the map $L\to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Then $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t. So an element $f(t)\in\mathcal{F}$

will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra, the study of which is the umbral calculus.

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{F}$ is the smallest integer k such that the coefficient of t^k does not vanish. Let $f(t), g(t) \in \mathcal{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then there exists a unique sequence of polynomials $S_n(x)$ (deg $S_n(x) = n$) such that

$$< g(t)f(t)^k | S_n(x) >= n!\delta_{n,k}, \text{ for } n, k \geq 0, \tag{2.1}$$

(cf. see [10], Theorem 2.3.1).

Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$.

It is an elementary fact that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{2.2}$$

where $\bar{f}(t)$ is the compositional inverse $f(t)$ satisfying $f(\bar{f}(t)) = t = \bar{f}(f(t))$.

For $S_n(x) \sim (g(t), f(t))$, the Sheffer identity is given by

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) P_{n-k}(y), \tag{2.3}$$

where $P_n(x) = g(t)S_n(x) \sim (1, f(t))$.

The following recurrence formula holds: for $S_n(x) \sim (g(t), f(t))$,

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{2.4}$$

For any $h(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$,

$$< h(t) | xp(x) >=< \partial_t h(t) | p(x) >, \tag{2.5}$$

$$< \tfrac{e^{yt}-1}{t} | p(x) >= \int_0^y p(u)du, \tag{2.6}$$

$$< e^{yt} | p(x) >= p(y), \tag{2.7}$$

$$e^{yt} p(x) = p(x+y). \tag{2.8}$$

$$\tag{2.9}$$

The following is the last one that we need: for $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$,

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{2.10}$$

with

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} \big(l(\bar{f}(t))\big)^k | x^n \right\rangle. \tag{2.11}$$

3. Some properties

From (1.1), we immediately see that

$$F_{n,\lambda}^{(r)}(x;y) \sim \big((1-y(e^t-1))^r, \tfrac{1}{\lambda}(e^{\lambda t}-1)\big), \tag{3.1}$$

and $\lim_{\lambda\to 0} F_{n,\lambda}^{(r)}(x;y) = F_n^{(r)}(x;y)$, where $F_n^{(r)}(x;y)$ are called two variable Fubini polynomials of order r and they are given by

$$\left(\frac{1}{1-y(e^t-1)}\right)^r e^{tx} = \sum_{n=0}^{\infty} F_n^{(r)}(x;y)\frac{t^n}{n!}. \tag{3.2}$$

Also, $\lim_{\lambda\to 0} F_{n,\lambda}^{(r)}(y) = F_n^{(r)}(y)$. $\lim_{\lambda\to 0} Ob_{n,\lambda}^{(r)}(x) = Ob_n^{(r)}(x)$, where $F_n^{(r)}(y)$ are called Fubini polynomials of order r with

$$\left(\frac{1}{1-y(e^t-1)}\right)^r = \sum_{n=0}^{\infty} F_n^{(r)}(y)\frac{t^n}{n!}, \tag{3.3}$$

and $Ob_n^{(r)}(x)$ are ordered Bell polynomials of order r with

$$\left(\frac{1}{2-e^t}\right)^r e^{tx} = \sum_{n=0}^{\infty} Ob_n^{(r)}(x)\frac{t^n}{n!}. \tag{3.4}$$

A degenerate version of the Stirling numbers of the second kind $S_2(n,k)$ are the degenrate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ given by

$$\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [5, 9])}. \tag{3.5}$$

Here we note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, and

$$S_{2,\lambda}(n,k) = \sum_{m=k}^{n} \lambda^{n-m} S_1(n,m) S_2(m,k), \quad \text{(see [5])}, \tag{3.6}$$

where $S_1(n,k)$ are the Stirling numbers of the first kind.

Let us consider the higher-order degenerate Fubini polynomials $F_{n,\lambda}^{(r)}(y)$.

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}(y)\frac{t^n}{n!} &= \big(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\big)^{-r} \\ &= \sum_{k=0}^{\infty}(r+k-1)_k y^k \frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k \\ &= \sum_{k=0}^{\infty}(r+k-1)_k y^k \frac{1}{k!}\sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k-1)_k S_{2,\lambda}(n,k) y^k\right)\frac{t^n}{n!}.\end{aligned}$$

Thus we obtain

$$F_{n,\lambda}^{(r)}(y) = \sum_{k=0}^{n}(r+k-1)_k S_{2,\lambda}(n,k) y^k, \tag{3.7}$$

and

$$F_{n,\lambda}^{(r)}(1) = Ob_{n,\lambda}^{(r)} = \sum_{k=0}^{n}(r+k-1)_k S_{2,\lambda}(n,k). \tag{3.8}$$

We claim that

$$\frac{1}{(1-y)^r}F_{n,\lambda}^{(r)}\big(\tfrac{y}{1-y}\big) = \sum_{k=0}^{\infty}\binom{r+k-1}{k}(k)_{n,\lambda} y^k, \tag{3.9}$$

where $(x)_{0,\lambda}=1$, and $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n\geq 1$. In particular, $y=\frac{1}{2}$ gives us

$$F_{n,\lambda}^{(r)}(1) = Ob_{n,\lambda}^{(r)} = \frac{1}{2^r}\sum_{k=0}^{\infty}\binom{r+k-1}{k}\frac{(k)_{n,\lambda}}{2^k}. \tag{3.10}$$

Also, from (2.3), (3.1) and (3.7), we see that

$$\begin{aligned}F_{n,\lambda}^{(r)}(x;y) &= \sum_{m=0}^{n}\binom{n}{m}F_{m,\lambda}^{(r)}(y)(x)_{n-m,\lambda} \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(r+k-1)_k S_{2,\lambda}(m,k)(x)_{n-m,\lambda} y^k,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned} F_{n,\lambda}^{(r)}(x;y) &= \sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda}^{(r-1)}(x;y) F_{n-m,\lambda}(y) \\ &= \sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda}^{(r-1)}(y) F_{n-m,\lambda}(x;y). \end{aligned} \tag{3.12}$$

Setting $x = 0$ in (3.12) yields

$$F_{n,\lambda}^{(r)}(y) = \sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda}^{(r-1)}(y) F_{n-m,\lambda}(y) \tag{3.13}$$

Now, from (3.7), (3.8), (3.10), and (3.11), we have the following result.

Theorem 3.1. *For $n \geq 0$, we have the following expressions.*

$$F_{n,\lambda}^{(r)}(x;y) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} (r+k-1)_k S_{2,\lambda}(m,k)(x)_{n-m,\lambda} y^k,$$

$$F_{n,\lambda}^{(r)}(y) = \sum_{k=0}^{n} (r+k-1)_k S_{2,\lambda}(n,k) y^k,$$

and

$$\begin{aligned} Ob_{n,\lambda}^{(r)} &= \sum_{k=0}^{n} (r+k-1)_k S_{2,\lambda}(n,k) \\ &= \frac{1}{2^r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} \frac{(k)_{n,\lambda}}{2^k}. \end{aligned}$$

Before proceeding to the next result, we recall here that the degenerate Frobenius-Euler polynomials $H_{n,\lambda}^{(r)}(u|x)$ of order r are defined by

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}(u|x) \frac{t^n}{n!}, \ (u \neq 1). \tag{3.14}$$

We observe now that, for $y \neq 0$,

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}(x;y) \frac{t^n}{n!} &= \left(\frac{1}{1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1)} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{1 - \frac{1+y}{y}}{(1+\lambda t)^{\frac{1}{\lambda}} - \frac{1+y}{y}} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}\left(\tfrac{1+y}{y} | x\right) \frac{t^n}{n!} \end{aligned}$$

Hence

$$F^{(r)}_{n,\lambda}(x;y) = H^{(r)}_{n,\lambda}\left(\tfrac{1+y}{y}|x\right), \ (y \neq 0). \tag{3.15}$$

Collecting (3.12),(3.13) and (3.15), we have the next theorem.

Theorem 3.2. *For $n \geq 0$, we have the following identities.*

$$\begin{aligned} F^{(r)}_{n,\lambda}(x;y) &= \sum_{m=0}^{n} \binom{n}{m} F^{(r-1)}_{m,\lambda}(x;y) F_{n-m,\lambda}(y) \\ &= \sum_{m=0}^{n} \binom{n}{m} F^{(r-1)}_{m,\lambda}(y) F_{n-m,\lambda}(x;y), \\ F^{(r)}_{n,\lambda}(y) &= \sum_{m=0}^{n} \binom{n}{m} F^{(r-1)}_{m,\lambda}(y) F_{n-m,\lambda}(y), \end{aligned}$$

and

$$F^{(r)}_{n,\lambda}(x;y) = H^{(r)}_{n,\lambda}\left(\tfrac{1+y}{y}|x\right), \ (y \neq 0).$$

The next discussion needs the following observation:

$$\begin{aligned} \left(1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1)\right)^{r} &= \sum_{l=0}^{\infty} (r)_l (-y)^l \frac{1}{l!} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^l \\ &= \sum_{l=0}^{\infty} (r)_l (-y)^l \sum_{k=l}^{\infty} S_{2,\lambda}(k,l) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} (r)_l S_{2,\lambda}(k,l)(-y)^l \right) \frac{t^k}{k!}. \end{aligned} \tag{3.16}$$

Now, from (1.1) and (3.16), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} = (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} (r)_l S_{2,\lambda}(k,l)(-y)^l \right) \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} F^{(r)}_{m,\lambda}(x;y) \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} (r)_l S_{2,\lambda}(k,l)(-y)^l F^{(r)}_{n-k,\lambda}(x;y) \right) \frac{t^n}{n!}. \end{aligned}$$

Thus we obtain

$$\begin{aligned}(x)_{n,\lambda} &= \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(r)_l S_{2,\lambda}(k,l)(-y)^l F^{(r)}_{n-k,\lambda}(x;y)\\ &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_{2,\lambda}(n-k,l)(-y)^l F^{(r)}_{k,\lambda}(x;y)\end{aligned} \tag{3.17}$$

Letting $x = 0$, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_{2,\lambda}(n-k,l)(-y)^l F^{(r)}_{k,\lambda}(y) = \begin{cases}1, & \text{for } n = 0,\\ 0, & \text{for } n \geq 1,\end{cases} \tag{3.18}$$

which is equivalent to the following (3.19).

$$\begin{aligned}&F^{(r)}_{0,\lambda}(y) = 1,\\ &F^{(r)}_{n,\lambda}(y) = -\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_{2,\lambda}(n-k,l)(-y)^l F^{(r)}_{k,\lambda}(y), \text{ for } n \geq 1.\end{aligned} \tag{3.19}$$

The next result following from (3.17) and (3.19).

Theorem 3.3. *For $n \geq 0$, we have*

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_{2,\lambda}(n-k,l)(-y)^l F^{(r)}_{k,\lambda}(x;y),$$

and

$$F^{(r)}_{n,\lambda}(y) = -\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_{2,\lambda}(n-k,l)(-y)^l F^{(r)}_{k,\lambda}(y), \text{ for } n \geq 1,$$

with $F^{(r)}_{0,\lambda}(y) = 1$.

Assume now that $n \geq 1$. Then, using (2.5) we have

$$\begin{aligned}F^{(r)}_{n,\lambda}(z;y) &= \left\langle \frac{(1+\lambda t)^{\frac{z}{\lambda}}}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r} \middle| x^n \right\rangle\\ &= \left\langle \left(\partial_t \frac{1}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r}\right)(1+\lambda t)^{\frac{z}{\lambda}} \middle| x^{n-1} \right\rangle\\ &\quad + \left\langle \frac{1}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r}\left(\partial_t(1+\lambda t)^{\frac{z}{\lambda}}\right) \middle| x^{n-1} \right\rangle.\end{aligned} \tag{3.20}$$

The seoncd term of (3.20) is clearly $zF^{(r)}_{n-1,\lambda}(z-\lambda;y)$. On the other hand, the first term of (3.20) is

$$ry\left\langle \frac{1}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^{r+1}}(1+\lambda t)^{\frac{z+1-\lambda}{\lambda}}\Big| x^{n-1}\right\rangle$$
$$= ryF^{(r+1)}_{n-1,\lambda}(z+1-\lambda;y).$$

Hence we have shown that

$$F^{(r)}_{n,\lambda}(z;y) = zF^{(r)}_{n-1,\lambda}(z-\lambda;y) + ryF^{(r+1)}_{n-1,\lambda}(z+1-\lambda;y). \tag{3.21}$$

We state (3.21) as the following theorem.

Theorem 3.4. *For $n \geq 0$, we have*

$$F^{(r)}_{n+1,\lambda}(x;y) = xF^{(r)}_{n,\lambda}(x-\lambda;y) + ryF^{(r+1)}_{n,\lambda}(x+1-\lambda;y),$$

and

$$F^{(r)}_{n+1,\lambda}(y) = ryF^{(r+1)}_{n,\lambda}(1-\lambda;y).$$

From (2.4) and (3.1), we note that

$$F^{(r)}_{n+1,\lambda}(x;y) = \left(x - \frac{g'(t)}{g(t)}\right) e^{-\lambda t} F^{(r)}_{n,\lambda}(x;y)$$
$$= xF^{(r)}_{n,\lambda}(x-\lambda;y) - e^{-\lambda t}\left(\frac{-rye^t}{1-y(e^t-1)}\right) F^{(r)}_{n,\lambda}(x;y)$$
$$= xF^{(r)}_{n,\lambda}(x-\lambda;y) + rye^{(1-\lambda)t}\frac{1}{1-y(e^t-1)} F^{(r)}_{n,\lambda}(x;y)$$
$$= xF^{(r)}_{n,\lambda}(x-\lambda;y) + ryF^{(r+1)}_{n,\lambda}(x+1-\lambda;y).$$

This gives another way of obtaining the result in Theorem 3.4.
Finally, we note the following.

$$
\begin{aligned}
&F_{n,\lambda}^{(r)}(z+1;y)-F_{n,\lambda}^{(r)}(z;y)\\
&=\left\langle \sum_{l=0}^{\infty}\left(F_{l,\lambda}^{(r)}(z+1;y)-F_{l,\lambda}^{(r)}(z;y)\right)\frac{t^l}{l!}\ \Big|\ x^n\right\rangle\\
&=\left\langle \frac{(1+\lambda t)^{\frac{z}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r}\ \Big|\ x^n\right\rangle\\
&=\frac{1}{y}\left\langle \frac{(1+\lambda t)^{\frac{z}{\lambda}}}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r}-\frac{(1+\lambda t)^{\frac{z}{\lambda}}}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^{r-1}}\ \Big|\ x^n\right\rangle\\
&=\frac{1}{y}\left(\left\langle \sum_{l=0}^{\infty}F_{l,\lambda}^{(r)}(z;y)\frac{t^l}{l!}\ \Big|\ x^n\right\rangle-\left\langle \sum_{l=0}^{\infty}F_{l,\lambda}^{(r-1)}(z;y)\frac{t^l}{l!}\ \Big|\ x^n\right\rangle\right)\\
&=\frac{1}{y}\left(F_{n,\lambda}^{(r)}(z;y)-F_{n,\lambda}^{(r-1)}(z;y)\right).
\end{aligned}
$$

Then we have derived the following identity:

$$
yF_{n,\lambda}^{(r)}(z+1;y)=(y+1)F_{n,\lambda}^{(r)}(z;y)-F_{n,\lambda}^{(r-1)}(z;y). \tag{3.22}
$$

By (3.22), we obtain the following result.

Theorem 3.5. *For $n\geq 0$ and $r\geq 2$, we have*

$$
yF_{n,\lambda}^{(r)}(x+1;y)=(y+1)F_{n,\lambda}^{(r)}(x;y)-F_{n,\lambda}^{(r-1)}(x;y).
$$

4. Some special polynomials in term of $F_{n,\lambda}^{(r)}(x;y)$

In this section, we will express some families of special polynomials as linear combinations of the two variable degenerate higher-order Fubini polynomials $F_{n,\lambda}^{(r)}(x;y)$. For this, as it turns out it is more convenient to use (2.1) than (2.11).

Let $p(x)\in\mathbb{C}[x]$ be of degree $\leq n$. Then we can write

$$
p(x)=\sum_{m=0}^{n}a_mF_{m,\lambda}^{(r)}(x;y),
$$

for unique $a_m\in\mathbb{C}(y)$.

We now note from (2.1) and (3.1) that

$$\begin{aligned} &\left\langle \left(1-y(e^t-1)\right)^r \left(\tfrac{1}{\lambda}(e^{\lambda t}-1)\right)^m \mid p(x)\right\rangle \\ &= \sum_{l=0}^{n} a_l \left\langle \left(1-y(e^t-1)\right)^r \left(\tfrac{1}{\lambda}(e^{\lambda t}-1)\right)^m \mid F_{l,\lambda}^{(r)}(x;y)\right\rangle \\ &= \sum_{l=0}^{n} a_l l! \delta_{m,l} \\ &= m! a_m. \end{aligned} \tag{4.1}$$

Further, From (4.1), we have

$$\begin{aligned} a_m &= \frac{1}{m!}\left\langle \left(1-y(e^t-1)\right)^r \left(\tfrac{1}{\lambda}(e^{\lambda t}-1)\right)^m \mid p(x)\right\rangle \\ &= \frac{1}{\lambda^m}\left\langle \left(1-y(e^t-1)\right)^r \mid \frac{1}{m!}(e^{\lambda t}-1)^m p(x)\right\rangle \\ &= \frac{1}{\lambda^m}\left\langle \left(1-y(e^t-1)\right)^r \mid \sum_{j=m}^{\infty} S_2(j,m)\frac{\lambda^j}{j!} t^j p(x)\right\rangle \\ &= \frac{1}{\lambda^m}\sum_{j=m}^{n} S_2(j,m)\frac{\lambda^j}{j!}\left\langle \left(1-y(e^t-1)\right)^r \mid t^j p(x)\right\rangle \\ &= \sum_{j=m}^{n} \frac{1}{j!} S_2(j,m)\lambda^{j-m}\left\langle \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l\right)\frac{t^k}{k!} \mid t^j p(x)\right\rangle \\ &= \sum_{j=m}^{n} \frac{1}{j!} S_2(j,m)\lambda^{j-m}\sum_{k=0}^{n-j}\frac{1}{k!}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l \left\langle 1 \mid t^{j+k}p(x)\right\rangle. \end{aligned} \tag{4.2}$$

We will use (4.2) throughout this section.
For $p(x) = B_n(x)$,

$$\begin{aligned} a_m &= \sum_{j=m}^{n} \frac{1}{j!} S_2(j,m)\lambda^{j-m}\sum_{k=0}^{n-j}\frac{1}{k!}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l (n)_{j+k} B_{n-j-k} \\ &= \sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l \lambda^{j-m} S_2(j,m) S_2(k,l) B_{n-j-k}(-y)^l. \end{aligned} \tag{4.3}$$

Let $H_n(u|x)$ be the Frobenius-Euler polynomials given by $\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(u|x)\frac{t^n}{n!}$, $(u \neq 1)$.

Similarly, for $p(x) = H_n(u|x)$,

$$a_m = \sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l\lambda^{j-m}S_2(j,m)S_2(k,l)H_{n-j-k}(u)(-y)^l, \tag{4.4}$$

where $H_n(u) = H_n(u|0)$ are called the Frobenius-Euler numbers.

On the other hand, for $p(x) = x^n$,

$$\begin{aligned} a_m &= \sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l\lambda^{j-m}S_2(j,m)S_2(k,l)(-y)^l\langle 1 \mid x^{n-j-k}\rangle \\ &= \sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l\lambda^{j-m}S_2(j,m)S_2(k,l)(-y)^l\delta_{n-j,k} \\ &= \sum_{j=m}^{n}\sum_{l=0}^{n-j}\binom{n}{j}(r)_l\lambda^{j-m}S_2(j,m)S_2(n-j,l)(-y)^l. \end{aligned} \tag{4.5}$$

Collecting the results in (4.3), (4.4), and (4.5), we obtain the next theorem.

Theorem 4.1. *For $n \geq 0$, we have*

$$\begin{aligned} B_n(x) &= \sum_{m=0}^{n}\left(\sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l\lambda^{j-m}S_2(j,m)S_2(k,l)B_{n-j-k}(-y)^l\right) \\ &\quad\times F_{m,\lambda}^{(r)}(x;y), \\ H_n(u|x) &= \sum_{m=0}^{n}\left(\sum_{j=m}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{k}\binom{n}{j}\binom{n-j}{k}(r)_l\lambda^{j-m}S_2(j,m)S_2(k,l)H_{n-j-k}(u)(-y)^l\right) \\ &\quad\times F_{m,\lambda}^{(r)}(x;y), \end{aligned}$$

and

$$x^n = \sum_{m=0}^{n}\left(\sum_{j=m}^{n}\sum_{l=0}^{n-j}\binom{n}{j}(r)_l\lambda^{j-m}S_2(j,m)S_2(n-j,l)(-y)^l\right)F_{m,\lambda}^{(r)}(x;y).$$

Applying (4.2) to $p(x) = Bel_n(x) = \sum_{i=0}^{n} S_2(n,i)x^i$, we obtain

$$a_m = \sum_{j=m}^{n}\frac{1}{j!}S_2(j,m)\lambda^{j-m}\sum_{k=0}^{n-j}\frac{1}{k!}\sum_{l=0}^{k}(r)_lS_2(k,l)(-y)^l\left\langle 1 \mid t^{j+k}Bel_n(x)\right\rangle. \tag{4.6}$$

Here

$$\begin{aligned}
&\left\langle 1 \mid t^{j+k} Bel_n(x) \right\rangle \\
&= \sum_{i=j+k}^{n} S_2(n,i) \left\langle 1 \,\middle|\, t^{j+k} x^i \right\rangle \\
&= \sum_{i=j+k}^{n} S_2(n,i)(i)_{j+k} \delta_{i,j+k} \\
&= S_2(n, j+k)(j+k)!.
\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we get

$$a_m = \sum_{j=m}^{n} \sum_{k=0}^{n-j} \sum_{l=0}^{k} \binom{j+k}{j} (r)_l \lambda^{j-m} S_2(j,m) S_2(k,l) S_2(n, j+k)(-y)^l. \tag{4.8}$$

Similarly, applying (4.2), to $p(x) = (x)_n = \sum_{i=0}^{n} S_1(n,i) x^i$, we have

$$a_m = \sum_{j=m}^{n} \sum_{k=0}^{n-j} \sum_{l=0}^{k} \binom{j+k}{j} (r)_l \lambda^{j-m} S_2(j,m) S_2(k,l) S_1(n, j+k)(-y)^l. \tag{4.9}$$

We now state (4.8) and (4.9) as a theorem.

Theorem 4.2. *For $n \geq 0$, we have*

$$\begin{aligned}
Bel_n(x) = \sum_{m=0}^{n} \Bigg(&\sum_{j=m}^{n} \sum_{k=0}^{n-j} \sum_{l=0}^{k} \binom{j+k}{j} (r)_l \lambda^{j-m} S_2(j,m) S_2(k,l) \\
&\times S_2(n, j+k)(-y)^l \Bigg) F_{m,\lambda}^{(r)}(x;y),
\end{aligned}$$

and

$$\begin{aligned}
(x)_n = \sum_{m=0}^{n} \Bigg(&\sum_{j=m}^{n} \sum_{k=0}^{n-j} \sum_{l=0}^{k} \binom{j+k}{j} (r)_l \lambda^{j-m} S_2(j,m) S_2(k,l) \\
&\times S_1(n, j+k)(-y)^l \Bigg) F_{m,\lambda}^{(r)}(x;y).
\end{aligned}$$

5. $F_{n,\lambda}^{(r)}(x;y)$ in terms of some special polynomials

Here we would like to express the two variable higher-order degenerate Fubini polynomials $F_{n,\lambda}^{(r)}(x;y)$ as linear combinations of some well-known families of special polynomials.

For this, we first recall from (3.1) that

$$F_{n,\lambda}^{(r)}(x;y) \sim \big((1-y(e^t-1))^r, \tfrac{1}{\lambda}(e^{\lambda t}-1)\big).$$

We let

$$F_{n,\lambda}^{(r)}(x;y) = \sum_{m=0}^{n} C_{n,m} S_m(x), \tag{5.1}$$

with

$$S_n(x) \sim \big(h(t), l(t)\big).$$

Then, from (2.11), we see that

$$C_{n,m} = \frac{1}{m!}\left\langle \frac{h(\frac{1}{\lambda}\log(1+\lambda t))}{\big(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\big)^r}\big(l(\tfrac{1}{\lambda}\log(1+\lambda t))\big)^m \middle| x^n \right\rangle. \tag{5.2}$$

Throughout this section, we are going to use (5.2). Let $S_n(x) = F_n^{(r)}(x;y) \sim \big((1-y(e^t-1))^r, t\big)$. Then

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\big\langle \big(\tfrac{1}{\lambda}\log(1+\lambda t)\big)^m \mid x^n \big\rangle \\ &= \frac{1}{\lambda^m}\big\langle \tfrac{1}{m!}(\log(1+\lambda t))^m \mid x^n \big\rangle \\ &= \frac{1}{\lambda^m}\Big\langle \sum_{k=m}^{\infty} S_1(k,m)\frac{\lambda^k t^k}{k!} \mid x^n \Big\rangle \\ &= \frac{1}{\lambda^m}\sum_{k=m}^{n} S_1(k,m)\frac{\lambda^k}{k!}\big\langle 1 \,| t^k x^n \big\rangle \\ &= \frac{1}{\lambda^m}\sum_{k=m}^{n} S_1(k,m)\frac{\lambda^k}{k!}(n)_k \delta_{n,k} \\ &= \lambda^{n-m} S_1(n,m). \end{aligned} \tag{5.3}$$

Then we obtain the following result from (5.3).

Theorem 5.1. *For $n \geq 0$, we have*

$$F_{n,\lambda}^{(r)}(x;y) = \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) F_m^{(r)}(x;y).$$

To proceed to the next result, we need to observe the following:

$$\begin{aligned}\Big(1-y\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)\Big)^{-r} &= \sum_{k=0}^{\infty}(-r)_k(-y)^k\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k \\ &= \sum_{k=0}^{\infty}(r+k-1)_k y^k\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(r+k-1)_k S_{2,\lambda}(l,k)y^k\right)\frac{t^l}{l!}.\end{aligned} \tag{5.4}$$

Next, we let $S_n(x)=(x)_{n,\lambda}\sim\big(1,\frac{1}{\lambda}(e^{\lambda t}-1)\big)$. Then

$$\begin{aligned}C_{n,m} &= \frac{1}{m!}\big\langle\big(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\big)^{-r}\,\big|\,t^m x^n\big\rangle \\ &= \binom{n}{m}\big\langle(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))^{-r}\,\big|\,x^{n-m}\big\rangle \\ &= \binom{n}{m}\Big\langle\sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(r+k-1)_k S_{2,\lambda}(l,k)y^k\right)\frac{t^l}{l!}\,\Big|\,x^{n-m}\Big\rangle \\ &= \binom{n}{m}\sum_{l=0}^{n-m}\sum_{k=0}^{l}(r+k-1)_k S_{2,\lambda}(l,k)y^k\binom{n-m}{l}\delta_{n-m,l} \\ &= \binom{n}{m}\sum_{l=0}^{n-m}(r+k-1)_k S_{2,\lambda}(n-m,k)y^k.\end{aligned} \tag{5.5}$$

The next result follows from (5.5).

Theorem 5.2. *For $n\geq 0$, we have*

$$F_{n,\lambda}^{(r)}(x;y)=\sum_{m=0}^{n}\left(\binom{n}{m}\sum_{l=0}^{n-m}(r+k-1)_k S_{2,\lambda}(n-m,k)y^k\right)(x)_{m,\lambda}.$$

Now, $S_n(x)=F_{n,\lambda}^{(s)}(x;y)\sim\big((1-y(e^t-1))^s,\frac{1}{\lambda}(e^{\lambda t}-1)\big)$. If $s>r$, then, using (3.16), we get

$$C_{n,m}=\binom{n}{m}\sum_{l=0}^{n-m}(s-r)_l S_{2,\lambda}(n-m,l)(-y)^l. \tag{5.6}$$

On the other hand, if $s<r$, then, from (5.4), we obtain

$$C_{n,m}=\binom{n}{m}\sum_{k=0}^{n-m}(r-s+k-1)_k S_{2,\lambda}(n-m,k)y^k. \tag{5.7}$$

From (5.6) and (5.7), we have the following theorem.

Theorem 5.3. *For $n \geq 0$, the following holds.*
For $s > r$, we have

$$F_{n,\lambda}^{(r)}(x;y) = \sum_{m=0}^{n} \left(\binom{n}{m} \sum_{l=0}^{n-m} (s-r)_l S_{2,\lambda}(n-m,l)(-y)^l \right) F_{m,\lambda}^{(s)}(x;y);$$

for $s < r$, we have

$$F_{n,\lambda}^{(r)}(x;y) = \sum_{m=0}^{n} \left(\binom{n}{m} \sum_{l=0}^{n-m} (r-s+l-1)_l S_{2,\lambda}(n-m,l) y^l \right) F_{m,\lambda}^{(s)}(x;y).$$

Let us now consider the degenerate Bernoulli polynomials $S_n(x) = \beta_n(x|\lambda) \sim \left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}, \frac{1}{\lambda}(e^{\lambda t}-1)\right)$.

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{t^m}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r} \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} \,\Big|\, x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r} t^m \,\Big|\, \frac{1}{n+1} x^{n+1} \right\rangle \\ &= \frac{1}{n+1}\binom{n+1}{m} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^r} t^m \,\Big|\, x^{n+1-m} \right\rangle \\ &= \frac{1}{n+1}\binom{n+1}{m} \left\langle \sum_{k=0}^{\infty} \left(F_{k,\lambda}^{(r)}(1;y) - F_{k,\lambda}^{(r)}(y)\right) \frac{t^k}{k!} \,\Big|\, x^{n+1-m} \right\rangle \\ &= \frac{1}{n+1}\binom{n+1}{m} \left(F_{n+1-m,\lambda}^{(r)}(1;y) - F_{n+1-m,\lambda}^{(r)}(y)\right). \end{aligned} \tag{5.8}$$

Now, (5.8) gives the following result.

Theorem 5.4. *For $n \geq 0$, we have*

$$F_{n,\lambda}^{(r)}(x;y) = \frac{1}{n+1} \sum_{m=0}^{n} \binom{n+1}{m} \left(F_{n+1-m,\lambda}^{(r)}(1;y) - F_{n+1-m,\lambda}^{(r)}(y)\right) \beta_m(x|\lambda).$$

Finally, we would like to consider the degenerate Frobenius-Euler polynomials $S_n(x) = H_{n,\lambda}(u|x) = H_{n,\lambda}^{(1)}(u|x) \sim \left(\frac{e^t-u}{1-u}, \frac{1}{\lambda}(e^{\lambda t}-1)\right)$, (see, (3.14)). Then

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\frac{1}{1-u}\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}-u}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^{r}} t^{m} \mid x^{n} \right\rangle \\
&= \frac{1}{1-u}\binom{n}{m}\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}-u}{\left(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^{r}} \mid x^{n-m} \right\rangle \\
&= \frac{1}{1-u}\binom{n}{m}\Big(F_{n-m,\lambda}^{(r)}(1;y)-uF_{n-m,\lambda}^{(r)}(y)\Big).
\end{aligned} \tag{5.9}
$$

Our last result follows from (5.9).

Theorem 5.5. *For $n \geq 0$, we have*

$$
F_{n,\lambda}^{(r)}(x;y) = \frac{1}{1-u}\sum_{m=0}^{n}\binom{n}{m}\left(F_{n-m,\lambda}^{(r)}(1;y)-uF_{n-m,\lambda}^{(r)}(y)\right)H_{m,\lambda}(u|x).
$$

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Department of Mathematics, Sogang University, Seoul, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: gwjang@kw.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: sura@kw.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr